

From: [Liu, Yi-Kai \(Fed\)](#)
To: (b) (6)
Subject: Re: ieee
Date: Wednesday, February 22, 2017 7:35:48 PM
Attachments: [cryptanalysis.bib](#)
[cryptanalysis.pdf](#)
[cryptanalysis.tex](#)

Hi Stephen,

Thanks for sending that! I went through the document and made a few changes:

- Section 3.5: Added a paragraph about search via quantum walk.

- I reworked the section on quantum superposition attacks. In particular, I added a sentence about Boneh and Zhandry's work on quantum-secure MACs. Also, I moved the whole section into the "research frontiers" part of the paper, at the end. (Perhaps it will help make up for the fact that the QAOA and HHL algorithms have little, if any, relevance to cryptanalysis...)

- Section 4.1: I decided to mention multilinear maps, and not Buchmann-Williams, because I think that's more recent and interesting.

- I updated the bibliography. In particular, I removed some of the references to older research papers, and added some references to survey articles. (In the bib file, I tried to comment out the references I didn't want, but bibtex didn't like that, so instead I had to delete those references, sorry about that.)

Cheers, and thanks very much!

--Yi-Kai

From: Stephen Jordan (b) (6)
Sent: Wednesday, February 22, 2017 12:14:54 PM
To: Liu, Yi-Kai (Fed)
Subject: Re: ieee

Hi Yi-Kai,

Here is the current version. We have a 6000 word limit and doing wc on the .tex file gives 5106 right now. So, we've got a little room for expansion, though not a lot.

Best regards,

Stephen

On Wed, Feb 22, 2017 at 10:56 AM, Liu, Yi-Kai (Fed) <yi-kai.liu@nist.gov<<mailto:yi-kai.liu@nist.gov>>> wrote:
Hi Stephen,

(b) (6)

I think the referee is making the point that the Buchmann-Williams cryptosystem was already broken by Hallgren's quantum algorithm from 2007, which worked in constant-degree number fields. Whereas the attack on Soliloquy required much more recent techniques, and it involved solving a different and more sophisticated problem, in a degree- n number field, with a principal ideal whose generator was unusually short. I like the referee's suggestion to mention that these recent attacks also apply to some constructions for multilinear maps and indistinguishability obfuscation.

I can try to fix this. Whenever it's convenient, could you send me the most recent version of the paper?

Sorry for not getting back to you, I was busy with the PQCrypto deadline yesterday, although in the end we decided we weren't ready to submit anything. :(

--Yi-Kai

From: Stephen Jordan (b) (6)
Sent: Monday, February 20, 2017 3:30:57 PM
To: Liu, Yi-Kai (Fed)
Subject: Re: ieee

(b) (6)

The main comment from the referees that I wasn't sure what to make of was the following comment about the Buchmann-Williams cryptosystem. Perhaps you understand this better than I do?

Section 4.1:

Buchman -> Buchmann

The reference to Buchmann-Williams is not perfect. It only exists in degree two, which was broken by Hallgren in JACM'07, or maybe the constant degree case, which was broken by Hallgren in STOC'05.

You could mention that the currently popular topic of indistinguishability obfuscation currently relies on the same assumption that is broken. Smart-Vercauteren FHE does too.

Best regards,

Stephen

On Mon, Feb 20, 2017 at 3:14 PM, Liu, Yi-Kai (Fed) <yi-kai.liu@nist.gov<<mailto:yi-kai.liu@nist.gov>><<mailto:yi-kai.liu@nist.gov>>>> wrote:
[Hi Stephen,](#)

[Oh, that's great. Thanks very much! I'll take a look and get back to you. I'm sorry for neglecting this --](#) (b) (6)

--Yi-Kai

From: Stephen Jordan (b) (6)
Sent: Monday, February 20, 2017 1:55:23 PM
To: Liu, Yi-Kai (Fed); Yi-Kai Liu
Subject: Re: ieee

[Actually, I just got a response from IEEE. They gave us an extension until Friday. So we have a little breathing](#)

room.

Best regards,

Stephen

On Mon, Feb 20, 2017 at 1:09 PM, Stephen Jordan (b) (6) <mailto:spj.jordan@gmail.com>><mailto:spj.jordan@gmail.com><mailto:spj.jordan@gmail.com>>>> wrote: HI Yi-Kai,

I have made an updated version of our IEEE manuscript. The main change is that I added a brief section about quantum superposition attacks. I have also removed the statements regarding the relative novelty of Kuperberg's sieve and van Dam's period finding algorithm which reviewer #2 found objectionable, and made the miscellaneous small grammar tweaks requested by referee #1.

Both referees complained that we didn't cite enough references. However, the solicitation from the magazine explicitly limits the number of references to 15, which we saturated. So, I mostly left that alone, although we now have 18 references due to three new ones in the new section.

Let me know what you think. I need to upload our revision sometime this afternoon. Looking at the emails we received from IEEE it appears to me that we were technically supposed to do it before today rather than during today. However, I was incapacitated by illness for the latter half of last week and forgot about it.

Best regards,

Stephen

Quantum Cryptanalysis: Shor, Grover, and Beyond

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Abstract

Peter Shor’s 1994 discovery of polynomial-time quantum algorithms for discrete logarithms and integer factorization showed that the currently used public-key cryptosystems will be insecure in an era of large-scale universal quantum computers. A number of promising public-key cryptosystems thought to be resistant to quantum attack have been proposed. These include lattice-based, code-based, and multivariate systems, as well as hash-based signatures. However, some recently-discovered quantum algorithms have resulted in complete breaks of cryptosystems previously expected to be secure against quantum attack. Although a number of strong candidates for post-quantum cryptography remain standing, it is clear that continued attention to quantum algorithms for cryptanalysis is warranted. Here we survey recent developments in quantum algorithms, focusing on those most likely to be of relevance to cryptanalysis.

1 Introduction

In 1994 Peter Shor discovered polynomial-time quantum algorithms for integer factorization and discrete logarithms. If sufficiently large-scale quantum computers are built, Shor’s algorithms will completely break the RSA cryptosystem and signature scheme, Diffie-Helman key exchange, and elliptic curve cryptography such as ECDH and ECDSA. At present, there are still no sufficiently large-scale quantum computers to perform cryptanalysis. However, there has been substantial progress since 1994 both on quantum computational hardware and algorithms. In response to accelerating progress on quantum hardware in the last few years, efforts have been intensifying to develop practical alternatives to the public key cryptographic methods currently in use. Such “post-quantum” cryptographic schemes base their security on hard computational problems, such as finding short vectors in high-dimensional lattices or solving systems of multivariate quadratic equations over finite fields, that are not known to be susceptible to attack by quantum computers. However, since 1994, a small but dedicated community of quantum algorithm researchers has continually discovered quantum

algorithms offering speedups for a widening variety of problems. Whether the various proposed post-quantum cryptosystems are truly safe from quantum attack remains an area of active research.

2 Quantum computation

Two of the most common misconceptions about quantum computers are that they can achieve exponential speedup for generic problems through exponential parallelism or that the only speedups achievable by quantum computers are for integer factorization using Shor's algorithm and searching using Grover's algorithm. The truth is far more interesting than either of these caricatures. In reality, to outperform classical algorithms, quantum algorithms rely on subtle interference effects among amplitudes representing exponentially many different computational paths. The quantum speedups discovered so far extend well beyond factoring and searching, but they much are more rare and precious than is suggested by the metaphor of exponential parallelism.

A key element of quantum mechanics, which underlies quantum computation, is the principle of superposition. Each distinguishable state of a quantum system has a corresponding amplitude, which is a complex number of magnitude at most one. Upon measurement, the system will be found in a given state with probability given by the squared magnitude of the associated amplitude. The idea of superposition is often illustrated using Schrödinger's famous thought experiment in which a cat is put into a state that simultaneously has equal amplitude to be alive and dead. However, to observe non-classical behavior arising from superposition requires very careful isolation from stray interactions with the environment. Under ordinary circumstances, at the macroscopic level, noise washes out quantum effects far too quickly for them to be noticeable.

In accordance with the superposition principle, the state of a quantum computer with n quantum bits of memory (qubits) is described by a list of 2^n amplitudes, one for each possible length- n bit string. These exponentially complicated states give some hint as to the origin of the power of quantum computers. A full description of the state of a quantum computer with only 80 qubits would already be too large to store on all the hard drives ever manufactured. However, this is not the whole story. To completely describe the state of an n -bit classical probabilistic computer one would also need to write down 2^n numbers, namely the probabilities associated with each of the 2^n bit strings. So, exponentially complicated states are not a uniquely quantum phenomenon! The crucial difference is that quantum amplitudes can be positive or negative¹, and thus can interfere constructively or destructively like waves, whereas probabilities are always nonnegative and simply add. Only by carefully designing quantum algorithms to take advantage of interference effects is it possible to achieve exponential speedup over classical computation.

Over the past 20 years, steady progress has been made in the quest to actually build quantum computers. A key obstacle is that the superpositions and interference effects nec-

¹More generally, quantum amplitudes can have arbitrary complex phases. However, it turns out that real positive and negative amplitudes suffice for universal quantum computation.

essary to achieve quantum computational speedup are extremely fragile and easily destroyed by imprecise control signals and stray influences from the environment. However, in 1996 it was shown theoretically that, if error per operation in a quantum computer could be brought below a fixed threshold, then it is possible to carry out arbitrarily long quantum computations reliably through the use of error-correcting codes. Recently, individual operations on small collections of qubits have been demonstrated in multiple quantum computing platforms, such as ion traps and superconducting circuits, that are nominally below this threshold. To bring the overhead associated with error correction down to tolerable levels for practical systems it will be necessary to decrease the error rates still further. Nevertheless, this achievement has been widely viewed as a significant milestone in the development of quantum computers.

At the same time as universal quantum computers are being developed, other labs are developing more specialized devices such as quantum simulators for mimicking materials that are hard to simulate on conventional supercomputers, and quantum annealers for solving discrete optimization problems. These specialized quantum computers have been scaled up much further than universal digital quantum computers. In particular, the current generation of D-wave's quantum annealer has over 1000 qubits, whereas current prototypes of universal quantum computers have only tens of qubits. However, quantum simulators and quantum annealers are single-purpose devices, unable to run, for example, Shor's algorithm, and do not have any known applications to cryptanalysis. The remainder of this article will focus only on algorithms for universal digital quantum computers.

3 Quantum algorithms

In classical computing we are used to describing computations at different levels of abstraction. At the bottom level we have elementary logic operations such as AND, OR, and NOT gates. At an intermediate level we talk of arithmetic operations, if statements, for loops, and so on. At a high level we talk of commonly used subroutines, such as searching, sorting, inverting matrices, and fast Fourier transforms. The same holds for quantum computation. What follows is a brief tour of quantum algorithms from a high-level perspective, emphasizing quantum-algorithmic primitives of relevance to cryptanalysis. For readers who are interested in a more thorough treatment of quantum computation and quantum algorithms, there are a number of excellent books and review articles [1, 2, 3, 4].

3.1 Reversible Computing

In the earliest days of quantum computation research the main question being asked was not whether quantum effects could be used for computational advantage but rather whether they posed a barrier to continued miniaturization of computers. In particular, the dynamics of closed quantum systems are fully reversible, that is, information cannot be erased. Consequently, logic gates such as the AND gate, which takes two bits of input and produces one bit of output, cannot be directly implemented in a quantum-coherent way. However, pioneering

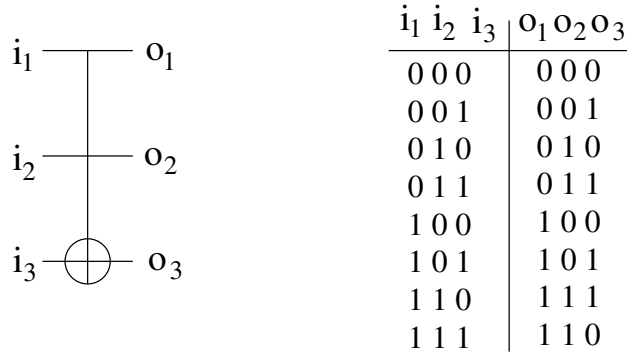


Figure 1: The Toffoli gate, illustrated above, is an example of a reversible gate; the input can be uniquely determined by the output. The Toffoli gate acts by flipping the value of the third (bottom) bit if and only if the first two bits are both one. If input i_3 is initialized to zero, then output o_3 contains i_1 AND i_2 . If input i_3 is initialized to one then o_3 contains i_1 NAND i_2 . NAND is a universal logic gate, and therefore Toffoli is as well.

works by Lecerf, Bennett, Toffoli, and Fredkin in the 60s, 70s, and 80s, showed how universal classical computing could be performed in a reversible way, such as by using universal sets of logic gates that implement reversible maps from their input bits to their output bits. Consequently, any classical subroutine can be recompiled into reversible logic gates and queried in quantum superposition within a quantum algorithm. This recompilation incurs an efficiency cost of only constant-factor overhead.

Many quantum algorithms are usefully phrased in terms of oracle (or “black-box”) problems. One is given access to some efficiently computable function, and one wishes to deduce some property of it by making queries. Grover search and period finding, discussed in detail below, are two examples of this. Reversible computing underlies these quantum algorithms in that the oracles are ultimately classical subroutines that have been recompiled into reversible gate sets so that they may be run on a quantum computer and queried on superpositions of inputs.

3.2 Period finding

One of the most important high-level primitives in quantum algorithm design is period finding. The period finding problem is to find the period of a periodic function f , given the ability to evaluate f on arbitrary inputs. That is, there is some s such that $f(x + s) = x$ for all x , and the goal is to find s . Access to the source code for f may make this problem easier. For example, if one examines the code for f and sees that it simply returns the residue of x modulo s , then it is trivial to read off that the period is s . However, in many situations, the computation of f is sufficiently complicated that there may be no better way to find the period than by treating f as a black-box and querying it at various x until the period can be found. Such a black-box framework is frequently a useful simplifying assumption in the

analysis of quantum algorithms.

On a quantum computer, it is possible to make queries to a black-box function in superposition. For some problems, such as period finding, this ability is greatly advantageous. For others, such as computing the overall parity of the set of outputs produced by f , quantum superposition queries yield very little advantage. If the period s of f is exponentially large then classically one requires exponentially many queries, on average, to find s . After a number of queries small compared to \sqrt{s} , one will not encounter any pair x, y such that $f(x) = f(y)$ and consequently one will have learned nothing about the period. In contrast, by making only polynomially many (in $\log s$) superposition queries, a quantum computer can learn the period s . This is done by computing the quantum Fourier transform — an analogue of the classical fast Fourier transform, which operates on quantum superposition states. This discovery, made by Peter Shor in 1994, formed the heart of his quantum algorithms for integer factorization and discrete logarithms.

Shor’s algorithms for factoring and discrete logarithms completely break the RSA cryptosystem and signature scheme, and Diffie-Hellman key exchange. Furthermore, the quantum algorithms for period finding can straightforwardly generalize to other domains. In the examples above, the inputs to x have been integers taken modulo N . However, if the inputs to x are thought of as elements of a finite Abelian group then the quantum algorithm for period finding still applies with little modification. The period may be described by a set of generators for a subgroup of the inputs which leave the value of f invariant. Quantum algorithms for this “Abelian hidden subgroup problem” can in particular be applied to the groups associated with elliptic curves, thereby breaking Elliptic Curve Diffie-Hellman (ECDH) key exchange and the Elliptic Curve Digital Signature Algorithm (ECDSA).

Encouraged by these successes, many researchers have attempted to devise quantum algorithms for “hidden subgroup problems” (HSP) over various non-Abelian groups, such as the symmetric group, the dihedral group, and the general linear group over a finite field. These HSP’s are connected to a number of other well-known problems that seem to be hard for classical computers, namely the graph isomorphism problem, the shortest vector problem in lattice-based cryptography, and certain problems underlying the security of multivariate cryptosystems. To date there has not been a breakthrough in solving any of these famous problems on a quantum computer. But there have been a number of interesting results, such as Kuperberg’s algorithm, which solves the dihedral HSP in subexponential time.

3.3 Searching

Like period finding, searching can be cast in a black-box query framework. For example, given a function $f : D \rightarrow \{0, 1\}$ on some finite set D , find an $x \in D$ such that $f(x) = 1$. The hardest instances of such a problem are when $f(x) = 1$ only for a unique x . Classically solving this requires $|D|$ queries in the worst case and $|D|/2$ queries on average. Lov Grover discovered in 1996 that quantum computers can solve this problem using $O(\sqrt{|D|})$ superposition queries. Earlier lower bounds showed that this is optimal, that is, fewer than order $\sqrt{|D|}$ quantum queries will not suffice.

Although stated abstractly, this primitive is easily adapted to many real-world problems. For example, suppose $g : D \rightarrow R$ is a one-way function, and you wish to invert it, that is, calculate $g^{-1}(r)$ for some $r \in R$. From g one can easily construct $f : D \rightarrow \{0, 1\}$ which outputs 1 if $g(x) = r$ and 0 otherwise. Such a function can be compiled into a reversible circuit that can be run on a quantum computer and queried in superposition. In this way Grover’s algorithm can perform brute-force inversion of one-way functions in $O(\sqrt{|D|})$ time, a quadratic improvement over classical brute-force. This has led to a rule of thumb for post-quantum symmetric-key cryptography, which advocates doubling key sizes (and thereby squaring the size of D) to maintain a given security level against attack by Grover search. However, this rule of thumb likely overestimates the advantage that can be gained from running Grover search in practice, since brute-force attacks are often run in parallel on large clusters of computers, and the Grover speedup is less significant in this setting. More detailed analysis has yielded more precise guidance on choice of key sizes in response to the speedups afforded by replacing search-like subroutines in state of the art classical cryptanalysis algorithms with Grover search [5, 6].

A quantum algorithm based on Grover search can also speed up collision finding, that is, finding $x, y \in D$ such that $f(x) = f(y)$. Classically, this has classical worst-case complexity of order $\sqrt{|D|}$ queries, which is achieved by the birthday paradox. On a quantum computer, collisions can be found using $O(|D|^{1/3})$ queries. However, the quantum algorithm for collision finding has large memory requirements which may pose challenges for developing quantum computers to find collisions in modern hash functions [7].

3.4 Hidden Shift

Hidden shift problems are another class of idealized oracle-query problems that have real-world implications. In a hidden shift problem, we are given oracle access to some function f , and we know that $f(x) = g(x+s)$ for some fixed known function g and unknown shift s . The goal is to find s by making queries to f . For the hardest instances of hidden shift, quantum computers can achieve only a quadratic speedup. For example, suppose the domain of f and g is the integers modulo N and let $g(0) = 1$ and $g(x) = 0$ for all $x \neq 0$. Then, finding s is exactly the Grover search problem, for which quantum computers can achieve only a quadratic speedup over classical computers.

However, for more structured instances of hidden shift, quantum algorithms have been discovered yielding exponential speedup. In particular, the Legendre symbol, for any prime p , is defined as follows.

$$\left(\frac{x}{p}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{p} \\ 1 & \text{if } x \equiv m^2 \pmod{p} \text{ for some } m \\ -1 & \text{otherwise} \end{cases}$$

The shifted Legendre symbol problem is, given oracle access to $f(x) = \left(\frac{x+s}{p}\right)$ for known p , to find s . It turns out that quantum computers can solve this problem in time that scales only polynomially in $\log p$, whereas it is believed that no classical algorithm can achieve this.

(In fact, the presumed hardness of this problem had been used as the basis for a proposed cryptographic random number generator.)

Another problem that has beyond-Grover quantum speedup is the hidden shift problem for injective functions. In this case, the number of quantum queries that are information-theoretically sufficient to extract the shift is only $\text{poly}(\log N)$. However, no quantum algorithm has been found to solve hidden shift in $\text{poly}(\log N)$ total runtime (which includes not just the queries but also pre- and post-processing). The quantum algorithm with the smallest total runtime for solving the injective hidden shift problem is Kuperberg’s sieve, which runs in $2^{O(\sqrt{N})}$ time. Subsequent work improved upon Kuperberg’s sieve by giving a quantum algorithm for the injective hidden shift problem that runs in $2^{O(\sqrt{N})}$ time and also uses only $\text{poly}(\log N)$ qubits of quantum memory.

In [8] a subexponential-time quantum algorithm for finding isogenies between ordinary (*i.e.* non-supersingular) elliptic curves was constructed through a nontrivial reduction to the injective hidden shift problem. Roughly speaking, such isogenies are maps from one elliptic curve to another which preserve the associated group structure. This quantum algorithm can be used to attack certain proposed public-key cryptosystems that base their security on the assumed difficulty of finding isogenies of ordinary elliptic curves. However, this quantum algorithm is not applicable to attacking the cryptosystems based on the difficulty of finding isogenies between supersingular elliptic curves.

3.5 Quantum Walks

A very different kind of computational speedup can be obtained using an algorithmic technique known as a quantum walk. Quantum walks are a technique for exploring a graph or network. They are defined by analogy with classical random walks, in different ways — there are several variants. As an illustration, we describe the continuous-time quantum walk, which is particularly simple.

In a classical random walk, a particle moves around a graph, hopping from one vertex to another, where each step is chosen at random, *i.e.*, if the particle currently resides at vertex v , it moves to one of the neighboring vertices of v , chosen at random. This defines a stochastic process, with a Markov transition matrix A . The continuous-time quantum walk is defined as the quantum dynamics that is generated by the Schrodinger equation, with the Hamiltonian given by A .

Quantum walks can be used to explore graphs that are exponentially large, provided that the list of neighbors of any given vertex can be computed efficiently on a quantum computer. Of course, classical random walks can also be used to explore large graphs. But the behavior of a quantum walk is strikingly different. For instance, a classical random walk converges to the stationary or equilibrium distribution, which is often spread over the whole graph. A quantum walk, on the other hand, is a reversible process, so it never converges. Instead, the quantum particle “hits” different parts of the graph, and then returns. Which parts of the graph get “hit” depends in part on the patterns of constructive and destructive interference that are created by the complex-valued wavefunction of the quantum particle.

This unusual behavior is advantageous for solving certain problems. A famous example

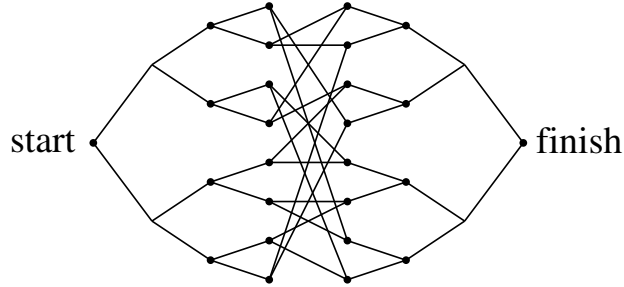


Figure 2: For glued trees, as illustrated above, a quantum walk algorithm can find the finish in time polynomial in the depth of the trees, whereas the fastest possible classical algorithm requires time polynomial in the number of vertices. Thus quantum walks achieve an exponential speedup for this problem.

is the problem of finding a path through two “glued trees.” Here, the graph consists of two full binary trees, whose leaves are connected by a random, alternating cycle. The problem is to find a path from the root of one tree to the root of the other tree.² This problem can be solved in polynomial time using a continuous-time quantum walk, but it requires exponential time using any classical algorithm. Intuitively, the reason is that any classical algorithm gets lost in the middle section of the graph, because there are exponentially many vertices, and exponentially few paths out of that section. The quantum walk, meanwhile, will propagate straight across the graph, due to constructive interference at each “level” of the tree.

The above algorithm is an intriguing example of an exponential quantum speedup that has nothing to do with the quantum Fourier transform, the hidden subgroup problem, or any kind of algebraic structure at all. This suggests that there may be other approaches to designing quantum algorithms for problems that are hard for classical computers.

In addition, while these exponential quantum speedups are somewhat rare, it is important to note that quantum walks can also achieve more modest polynomial speedups over classical computers, for a wide variety of search problems [9]. This gives a general-purpose technique for speeding up many classical cryptanalytic attacks.

4 Some Current Frontiers in Quantum Algorithms and Cryptanalysis

Quantum algorithms research is a highly technical subject in which major new algorithmic techniques are developed only once or twice per decade. Some of the most recent such developments are a quantum algorithm for the principal ideal problem, which has applications to lattice-based cryptography, a new method for solving discrete optimization problems on

²This is formulated as an oracle problem: one is given the name of the vertex at the root of the first tree, and an oracle that computes the neighbors of any vertex in this graph, and one is asked to discover the name of the vertex at the root of the other tree.

quantum computers called the Quantum Approximate Optimization Algorithm (QAOA), and a method for solving exponentially large systems of linear equations under certain conditions called the Harrow-Hassidim-Lloyd (HHL) algorithm. It is not yet clear what, if any, implications these primitives may have for quantum cryptanalysis.

4.1 Quantum Algorithms for Lattice Problems

There have been many attempts to devise quantum algorithms for finding short vectors in high-dimensional lattices. Early on, researchers noted a number of tantalizing connections between the mathematical methods used in lattice-based cryptography, and those used in quantum algorithms. In particular, many of the security proofs for lattice-based cryptosystems make use of random samples from certain periodic distributions over \mathbb{R}^n , as well as the Fourier transforms of these periodic distributions. Could there be a way of using the quantum Fourier transform to prepare the quantum superposition states corresponding to these distributions? These superposition states, sometimes called “quantum samples,” contain more information than classical random samples, and could be used to solve lattice problems.

In fact, this line of thinking did not lead to quantum algorithms for lattice problems. Instead, this idea was used by Oded Regev to prove one of the strongest security guarantees for lattice-based cryptography [10]. Specifically, Regev showed that breaking a certain public key encryption scheme is at least as hard as solving an intractable lattice problem, on a *quantum* computer. And so in this case, a technique originally intended for quantum cryptanalysis was ultimately used to provide evidence that a cryptosystem is actually secure.

More recently, however, there has been more progress on the side of quantum cryptanalysis. Specifically, there has been progress in developing quantum algorithms for solving problems involving lattices that have algebraic structure. (Lattices with algebraic structure are often used in cryptography, as they have more compact descriptions than general lattices. This makes the resulting cryptosystems more efficient.)

Recently, researchers have discovered efficient quantum algorithms for finding short generators of certain principal ideals in cyclotomic rings [11, 12]. These quantum algorithms demonstrate an exponential speedup over the fastest classical algorithms, as well as a potential security weakness of certain algebraically-structured lattices as compared to general lattices. In particular, these quantum algorithms break certain cryptosystems that use principal ideal lattices with unusually short generators, such as SOLILOQUY, and certain candidate constructions for multilinear maps. Other cryptosystems based on ideal lattices, such as NTRUEncrypt and ring-LWE schemes, are not broken by any presently-known quantum algorithm³.

The underlying technique used in these quantum algorithms is period-finding, but over a continuous (rather than discrete) domain. For functions on continuous domains, such as the real numbers, the problem of finding (approximate) periodicities becomes more subtle.

³See [13] for a very nice journalistic description of recent developments in the quantum cryptanalysis of lattice-based cryptosystems.

Nevertheless, quantum algorithms have been discovered which achieve exponential speedup over classical algorithms for this task. This in turn leads to polynomial-time quantum algorithms for various number theoretic problems, such as solving Pell’s equation, finding generators of principal ideals, and computing the class groups and unit groups of algebraic number fields, in time polynomial in the degree of the number field [14, 15].

4.2 Quantum Approximate Optimization Algorithm

QAOA was proposed in 2014 as a class of quantum algorithms for finding approximate solutions to discrete optimization problems such as MAX3SAT [16]. Such problems have wide-ranging practical applications. Furthermore, QAOA may be implementable on near-term quantum computing hardware that is not yet a universal error-corrected quantum computer. (QAOA shares these features with adiabatic quantum computation and quantum annealing to which it is closely related.) As a consequence, QAOA has attracted substantial interest from researchers since its introduction.

The difficulty of solving discrete optimization problems such as MAX3SAT varies greatly between instances. Consequently, proving meaningful bounds on the runtime of quantum or classical algorithms for these problems is extremely difficult. Furthermore, finding fully optimal solutions to these problems is NP-complete, and most computer scientists expect this task to have exponential complexity for worst case instances on both quantum and classical computers. However, in 2014 it was proven that QAOA can in polynomial-time find approximate solutions to a discrete optimization problem called Max E3LIN2, at an approximation scale that no polynomial-time classical algorithm had achieved. Prompted by this discovery, research intensified on classical approximation algorithms for Max E3LIN2, and in 2015 a classical algorithm provably outperforming QAOA on this problem was obtained. Nevertheless, the performance of QAOA on various discrete optimization problems remains a topic of intense research interest, both in terms of provable worst-case runtime bounds, and as a heuristic quantum algorithm.

4.3 Solving Linear Systems

Algorithms for solving linear algebra problems are among the most widely used primitives in all of scientific computing. Thus the 2009 announcement by Harrow, Hassidim, and Lloyd (HHL) of a quantum algorithm for solving exponentially large systems of linear equations in polynomial time was met with great excitement [17]. Specifically, given an $N \times N$ matrix A and an N -dimensional vector \vec{b} , over the real or complex numbers, the HHL algorithm can under certain circumstances construct in time $\text{poly}(\log N)$ a quantum state proportional to \vec{x} , the solution to $A\vec{x} = \vec{b}$. Subsequently, measurements on this quantum state can yield partial information about the solution \vec{x} . Simply reading all the entries of an $N \times N$ matrix takes time of order N^2 and no algorithm (quantum or classical) that uses N^2 time to read its input can possibly have total running time of only $\text{poly}(\log N)$. Instead, the HHL algorithm is of interest in the case that A is given implicitly by a subroutine which, when queried with a row index $j \in \{1, 2, \dots, N\}$, outputs a list of its nonzero matrix elements and their

locations. Furthermore, for the HHL algorithm to work efficiently, \vec{b} must be efficiently preparable as a quantum state. When this is the case, and A is highly sparse, *i.e.* each row has only $\text{poly}(\log N)$ nonzero entries, and has condition number at most $\text{poly}(\log N)$, it is possible for the HHL algorithm to construct a quantum state proportional to the solution \vec{x} in polynomial time even when N is exponentially large.

The problem solved by the HHL algorithm is significantly different from the problem solved by most conventional classical linear algebra packages; the input matrix is given by an oracle rather than explicitly stored, and the output is in the form of a quantum state. Furthermore, exponential speedup is only achievable if the condition number (*i.e.* ratio of largest to smallest eigenvalue) of the input matrix scales as a power of the logarithm of the dimension of the matrix. In short, this quantum algorithm is highly powerful but its applicability has limitations of an unfamiliar type. At present the boundaries of what can and cannot be achieved by quantum algorithms based on the HHL primitive are still being mapped out. Potential applications explored so far range from machine learning to calculating electromagnetic scattering crosssections.

4.4 Quantum Superposition Attacks

A recent line of work has focused on quantum superposition attacks, where an adversary prepares quantum superposition states and feeds them into parts of a cryptosystem that normally handle classical data. Such attacks may seem like an idea from a science fiction movie. But in certain scenarios, these attacks can be quite feasible. For instance, in the case of a fixed hash function such as SHA-3, the complete description of the function is public, and there is no secret key. So an adversary who has a quantum computer can, in fact, run the SHA-3 function on a superposition of inputs. A similar situation arises in the case of public key encryption, where an adversary can use the public key to encrypt a superposition of different messages. In other scenarios, these attacks may be less realistic, but nonetheless they raise fundamental questions about the foundations of cryptography in a quantum world.

These quantum superposition attacks can have surprising consequences. For example, some *symmetric-key* cryptographic primitives, such as 3-round Feistel ciphers, can be completely broken using quantum queries, by adapting Simon’s algorithm for solving the hidden subgroup problem over \mathbb{Z}_2^n [18, 19, 20]. Furthermore, certain message authentication codes, which rely on pairwise independent hash functions, can be broken using a quantum algorithm for polynomial interpolation [21].

5 Conclusion

Shor’s polynomial-time quantum algorithms for discrete logarithms and integer factorization imply that the currently widely-deployed public-key cryptosystems will be completely insecure in an era of large-scale universal quantum computers. A number of promising public-key cryptosystems hoped to be resistant to quantum attack have been proposed. These include lattice-based, code-based, and multivariate systems, as well as hash-based signatures. Many

of the classical attacks on these schemes, as well as on symmetric-key cryptography, can be sped up slightly through judicious use of Grover’s search algorithm. Such quantum attacks do not preclude use of these schemes, although they may dictate increased key sizes. However, there have also been a significant number of quantum algorithms discovered subsequent to Shor’s 1994 discovery of a quantum factoring algorithm and Grover’s 1996 discovery of a quantum search algorithm. Some of these recent breakthroughs in quantum algorithms have resulted in complete breaks (*e.g.* polynomial-time key recovery attacks) on cryptosystems that were previously expected to be secure against quantum computers. A clear lesson is that continued attention to quantum algorithms research will be crucial to the development of the next generation of public-key cryptosystems.

References

- [1] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- [2] Andrew M. Childs and Wim van Dam. Quantum algorithms for algebraic problems. *Rev. Mod. Phys.*, 82:1–52, 2010.
- [3] Stephen P. Jordan. Quantum algorithms zoo. <http://math.nist.gov/quantum/zoo/>, 2011.
- [4] Ashley Montanaro. Quantum algorithms: an overview. *npj Quantum Information*, 2(15023), 2016.
- [5] T. Laarhoven, M. Mosca, and J. van de Pol. Solving the shortest vector problem in lattices faster using quantum search. In *Proceedings of PQCrypto13*, pages 83–101, 2016.
- [6] Scott Fluhrer. Quantum cryptanalysis of NTRU. *IACR Cryptology ePrint Archive*, 2015:676, 2015. <https://eprint.iacr.org/2015/676>.
- [7] Daniel J. Bernstein. Cost analysis of hash collisions: Will quantum computers make SHARCS obsolete? In *SHARCS’09 Proceedings of the 4th Workshop on Special-purpose Hardware for Attacking Cryptographic Systems*, pages 105–116, 2009. <https://cr.yp.to/hash/collisioncost-20090517.pdf>.
- [8] Andrew M. Childs, David Jao, and Vladimir Soukharev. Constructing elliptic curve isogenies in quantum subexponential time. *Journal of Mathematical Cryptology*, 8(1):1–29, 2014. arXiv:1012.4019.
- [9] Frederic Magniez, Ashwin Nayak, Jeremie Roland, and Miklos Santha. Search via quantum walk. *SIAM J. Comput.*, 40(1):142–164, 2011.
- [10] Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore, MD, USA, May 22-24, 2005*, pages 84–93, 2005.

- [11] Peter Campbell, Michael Groves, and Dan Shepherd. SOLILOQUY: a cautionary tale. http://docbox.etsi.org/Workshop/2014/201410_CRYPT0/S07_Systems_and_Attacks/S07Groves_Annex.pdf, 2014.
- [12] Ronald Cramer, Leo Ducas, Chris Peikert, and Oded Regev. Recovering short generators of principal ideals in cyclotomic rings. In *Advances in Cryptology - EUROCRYPT 2016 - 35th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Vienna, Austria, May 8-12, 2016, Proceedings, Part II*, pages 559–585, 2016.
- [13] Natalie Wolchover. A tricky path to quantum-safe encryption. *Quanta Magazine*, 2015. <https://www.quantamagazine.org/20150908-quantum-safe-encryption/>.
- [14] Kirsten Eisenträger, Sean Hallgren, Alexei Kitaev, and Fang Song. A quantum algorithm for computing the unit group of an arbitrary degree number field. In *Proceedings of the ACM Symposium on Theory of Computing (STOC)*, pages 293–302, 2014.
- [15] Jean-Francois Biasse and Fang Song. Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 893–902, 2016.
- [16] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. A quantum approximate optimization algorithm. *arXiv:1411.4028*, 2014.
- [17] Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for solving linear systems of equations. *Physical Review Letters*, 15(103):150502, 2009. arXiv:0811.3171.
- [18] Hidenori Kuwakado and Masakatu Morii. Quantum distinguisher between the 3-round Feistel cipher and the random permutation. In *IEEE International Symposium on Information Theory (ISIT)*, pages 2682–2685, 2010.
- [19] Mark Kaplan, Gaëtan Leurent, Anthony Leverrier, and María Naya-Plasencia. Breaking symmetric cryptosystems using quantum period finding. *arXiv:1602.05973*, 2016.
- [20] Gorjan Alagic and Alexander Russell. Quantum-secure symmetric-key cryptography based on hidden shifts. *arXiv:1610.01187*, 2016.
- [21] Dan Boneh and Mark Zhandry. Quantum-secure message authentication codes. In *Advances in Cryptology - EUROCRYPT 2013, 32nd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Athens, Greece, May 26-30, 2013. Proceedings*, pages 592–608, 2013.